COMPOSITION OPERATORS WITHIN SINGLY GENERATED COMPOSITION C^*-ALGEBRAS

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Abstract. Let φ be a linear-fractional self-map of the open unit disk $\mathbb{D}$, not an automorphism, such that $\varphi(\zeta) = \eta$ for two distinct points ζ, η in the unit circle $\partial\mathbb{D}$. We consider the question of which composition operators lie in $C^*(C_\varphi, \mathcal{K})$, the unital C^*-algebra generated by the composition operator C_φ and the ideal $\mathcal{K}$ of compact operators, acting on the Hardy space H^2. This necessitates a companion study of the unital C^*-algebra generated by the composition operators induced by all parabolic non-automorphisms with common fixed point on the unit circle.

1. Introduction

Given any analytic self-map φ of the unit disk $\mathbb{D}$ in the complex plane, one can form the composition operator $C_\varphi : f \to f \circ \varphi$, which acts as a bounded operator on the Hardy space H^2. This paper is the second in a series of three investigating spectral theory in C^*-algebras generated by certain composition and Toeplitz operators. In the first article [15], we studied $C^*(T_z, C_\varphi)$, the unital C^*-algebra generated by the unilateral shift T_z on H^2 and a single composition operator C_φ with φ satisfying

(1)
$$\begin{cases} \varphi \text{ is a linear-fractional self-map of } \mathbb{D} \text{ which is not an automorphism, and} \\ \varphi(\zeta) = \eta \text{ for distinct points } \zeta, \eta \text{ in the unit circle } \partial\mathbb{D}. \end{cases}$$

Throughout the current paper, φ will always have this meaning. The algebra $C^*(T_z, C_\varphi)$ necessarily contains the ideal $\mathcal{K}$ of compact operators on H^2. The main result of [15] identifies $C^*(T_z, C_\varphi)/\mathcal{K}$ with a certain C^*-algebra of 2×2 matrix valued functions; see Theorem 4.12 of [15]. The case where φ is replaced by an automorphism of $\mathbb{D}$, or even a discrete group of automorphisms, has been studied by Jury [12], [13], and has a rather different character.

The shift T_z does not appear to play a role in the questions we consider in this paper; accordingly we omit it and study $C^*(C_\varphi, \mathcal{K})$, the unital C^*-algebra generated by C_φ, for φ as described above, and the compact operators. The composition $\varphi \circ \varphi$ has sup-norm strictly less than 1, so that $C_\varphi^2 = C_{\varphi\circ\varphi}$ is compact and non-zero. Since φ has no boundary fixed point, a theorem of Guyker [10] shows that C_φ is irreducible if and only if $\varphi(0) \neq 0$. It follows that when $\varphi(0) \neq 0$ the unital C^*-algebra $C^*(C_\varphi)$ generated by C_φ alone contains $\mathcal{K}$; see [5], p.74. We want our C^*-algebras to always contain $\mathcal{K}$, and we indicate this by continuing to write $C^*(C_\varphi, \mathcal{K})$ if the irreducibility criterion $\varphi(0) \neq 0$ is in doubt.

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Let $\mathcal{P}$ denote the dense subalgebra of $C^*(C_\varphi, \mathcal{K})$ consisting of all finite linear combinations of the identity I, all words in C_φ and C_φ^*, and all compact operators. An element B in $\mathcal{P}$ has a unique representation of the form

$$B = cI + f(C_\varphi^* C_\varphi) + g(C_\varphi C_\varphi^*) + C_\varphi p(C_\varphi^* C_\varphi) + C_\varphi^* q(C_\varphi C_\varphi^*) + K \tag{2}$$

where f, g, p and q are polynomials, $f(0) = 0 = g(0)$, c is complex, and K is compact. Let $s = 1/|\varphi'(\zeta)|$ and write $\mathcal{D}$ for the C^*-algebra of continuous 2×2 matrix-valued functions F on $[0, s]$ with $F(0)$ a scalar multiple of the identity, equipped with the supremum operator norm. It was shown in [15] that there is a unique *-homomorphism Ψ of $C^*(C_\varphi, \mathcal{K})$ onto $\mathcal{D}$ with Ker $\Psi = \mathcal{K}$ and such that

$$\Psi(B) = \begin{bmatrix} c+g & rp \\ rq & c+f \end{bmatrix} \tag{3}$$

where B is given by Equation (2) and $r(t) = \sqrt{t}$. Equivalently, we have a short exact sequence of C^*-algebras

$$0 \to \mathcal{K} \xrightarrow{i} C^*(C_\varphi, \mathcal{K}) \xrightarrow{\Psi} \mathcal{D} \to 0$$

where i is inclusion. For any operator T on H^2 we write $\|T\|_e$ for the essential norm of T, that is, the distance from T to the ideal $\mathcal{K}$. We note that if T is in $C^*(C_\varphi, \mathcal{K})$, then $\|T\|_e = \|\Psi(T)\|$.

For bounded operators A and B on H^2, let us write $A \equiv B \pmod{\mathcal{K}}$ if there exists a compact operator K with $A = B + K$. In [15] the authors used C. Cowen's well-known adjoint formula [7] to show that if

$$\psi(z) = \frac{az+b}{cz+d}$$

is a linear-fractional self map of $\mathbb{D}$, not an automorphism but satisfying $|\psi(z_0)| = 1$ for some $z_0 \in \partial\mathbb{D}$, then

$$C_\psi^* \equiv \frac{1}{|\psi'(z_0)|} C_{\sigma_\psi} \pmod{\mathcal{K}} \tag{4}$$

where σ_ψ is the so-called "Krein adjoint" of ψ,

$$\sigma_\psi(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

We denote the Krein adjoint of φ itself simply by σ, which the reader can distinguish by context from the spectrum $\sigma(T)$ or essential spectrum $\sigma_e(T)$ of an operator T.

Now consider an operator B in the dense subalgebra $\mathcal{P}$, as expressed in Equation (2). Since $I = C_z, C_\varphi^* C_\varphi \equiv s C_{\varphi\circ\sigma} \pmod{\mathcal{K}}$ and $C_\varphi C_\varphi^* \equiv s C_{\sigma\circ\varphi} \pmod{\mathcal{K}}$, where $s = 1/|\varphi'(\zeta)|$, we can rewrite Equation (2) as

$$B = cC_z + A_1 + A_2 + A_3 + A_4 + K' \tag{5}$$

where K' is compact and A_1, A_2, A_3, and A_4 are finite linear combinations of composition operators whose associated self-maps are chosen, respectively, from the four lists

$$\{(\varphi\circ\sigma)_{n_1}\}, \{(\sigma\circ\varphi)_{n_2}\}, \{(\varphi\circ\sigma)_{n_3}\circ\varphi\}, \{(\sigma\circ\varphi)_{n_4}\circ\sigma\}. \tag{6}$$

Here we write $(\psi)_n$ for the n^{th} iterate of a self-map ψ of $\mathbb{D}$, and let n_k range over the positive integers for $k = 1, 2$, and over the non-negative integers for $k = 3, 4$. See [15] for further details. Thus $C^*(C_\varphi, \mathcal{K})$ is spanned, modulo the compacts, by

actual composition operators. This leads to our main question: for which analytic self-maps ψ of $\mathbb{D}$ does C_ψ lie in $C^*(C_\varphi, \mathcal{K})$?

In particular we will describe explicitly, in both function-theoretic and operator-theoretic terms, all linear-fractional composition operators lying in $C^*(C_\varphi, \mathcal{K})$. This description plays a role in the third paper of our series [16] which is devoted to spectral theory in Toeplitz-composition algebras with several generators. Along the way here we show that if $C^*(\mathbb{P}_\gamma)$ denotes the unital C^*-algebra generated by composition operators induced by the parabolic non-automorphisms fixing γ in the unit circle, then there is a short exact sequence

$$0 \to \mathcal{K} \to C^*(\mathbb{P}_\gamma) \to C([0,1]) \to 0$$

of C^*-algebras, where $C([0,1])$ denotes the algebra of continuous functions on the unit interval.

2. Necessary conditions

If $\psi : \mathbb{D} \to \mathbb{D}$ is analytic we write $F(\psi)$ for the set of points α in $\partial\mathbb{D}$ at which ψ has a finite angular derivative in the sense of Caratheodory; see [8],[21]. In particular, if α is in $F(\psi)$, the nontangential limit $\psi(\alpha)$ necessarily exists and has modulus one. We write $\psi'(\alpha)$ for the angular derivative of ψ at α. Recall that this is the ordinary derivative if ψ extends analytically to a neighborhood of α and $|\psi(\alpha)| = 1$.

It is well known that if C_ψ is compact on H^2, then $F(\psi)$ is empty [20]. When C_ψ is considered as acting on the Bergman space in $\mathbb{D}$, the converse assertion is true [17]. On the space H^2 considered here, however, "C_ψ is compact" is a strictly stronger requirement than "$F(\psi)$ is empty"; see, for example, [8],[21]. Our first goal is to show that when C_ψ lies in $C^*(C_\varphi, \mathcal{K})$ these two conditions are equivalent.

First we recall that a linear-fractional self-map ρ of $\mathbb{D}$ is *parabolic* if ρ fixes one point γ in $\partial\mathbb{D}$ and is conjugate, via the map $(\gamma+z)/(\gamma-z)$, to translation in the right half-plane $\Omega = \{w : \operatorname{Re} w > 0\}$ by a complex number a with non-negative real part. We denote this parabolic map by ρ_a, or by $\rho_{\gamma,a}$ if the fixed point γ is not clear from the context. A linear-fractional map ρ with fixed point γ in $\partial\mathbb{D}$ is parabolic provided $\rho'(\gamma) = 1$. Another representation of ρ_a will prove useful. The map $\tau_\gamma(z) = i(\gamma - z)/(\gamma + z)$ carries $\mathbb{D}$ onto the upper half-plane $\{w : \operatorname{Im} w > 0\}$ and takes γ to 0, rather than infinity. We write u for the conjugate of ρ_a by τ_γ: $u = \tau_\gamma \circ \rho_a \circ \tau_\gamma^{-1}$. One readily computes that $u(w) = iw/(i + wa)$, and so

$$u''(0) = 2ia. \tag{7}$$

Also important for us will be several lower bounds for the essential norm of a linear combination of composition operators. Given an analytic self-map ψ of $\mathbb{D}$ and α in $F(\psi)$, we call $D_1(\psi, \alpha) \equiv (\psi(\alpha), \psi'(\alpha))$ the *first order data vector* for ψ at α. If we have a finite collection of maps $\psi_1, \psi_2 \cdots, \psi_n$ and α lies in the union of the finite angular derivative sets $F(\psi_1), F(\psi_2), \cdots F(\psi_n)$, we define

$$\mathcal{D}_1(\alpha) = \{D_1(\psi_j, \alpha) : 1 \le j \le n \text{ and } \alpha \in F(\psi_j)\},$$

the set of possible first order data vectors at α coming from that collection of maps. Theorem 5.2 in [14] states that if $\mathcal{D}_1(\alpha)$ is non-empty, then for any complex numbers $c_1, \cdots, c_n$,

$$\|c_1C_{\psi_1}+\cdots+c_nC_{\psi_n}\|_e^2 \geq \sum_{\mathbf{d}\in\mathcal{D}_1(\alpha)} \left| \sum_{\substack{\alpha\in F(\psi_j)\\ D_1(\psi_j,\alpha)=\mathbf{d}}} c_j \right|^2 \frac{1}{|d_1|}, \tag{8}$$

where $\mathbf{d}=(d_0,d_1)$.

There is a higher order version of this lower bound which works provided that for the specific α in $F(\psi)$, ψ is analytically continuable across $\partial\mathbb{D}$ at α and $|\psi(e^{i\theta})|<1$ for $e^{i\theta}$ near to, but not equal to, α. More detail can be found in [14], where a somewhat larger class of maps is considered. For such α, the curve $\psi(e^{i\theta})$, $e^{i\theta}$ near α, automatically has positive and even order of contact $2m$ with $\partial\mathbb{D}$ when $e^{i\theta}=\alpha$; that is,

$$\frac{1-|\psi(e^{i\theta})|^2}{|\psi(\alpha)-\psi(e^{i\theta})|^{2m}}$$

is bounded above and away from 0 for $e^{i\theta}$ near α. For $k\geq 1$ the k^{th} order data vector

$$D_k(\psi,\alpha)\equiv(\psi(\alpha),\psi'(\alpha),\cdots,\psi^{(k)}(\alpha))$$

makes sense. Given a finite collection $\psi_1\cdots,\psi_n$ of such maps and $k\geq 2$, we write $\mathcal{M}_k(\alpha)$ for the set of integers j, $1\leq j\leq n$, for which $F(\psi_j)$ contains α and the order of contact of ψ_j at α is at least k. Further, put

$$\mathcal{D}_k(\alpha)=\{D_k(\psi_j,\alpha): j\in\mathcal{M}_k(\alpha)\},$$

the set of possible k^{th} order data vectors at α for associated orders of contact at least k. With this notation, Theorem 5.7 in [14] states that for any $k\geq 2$ and complex constants $c_1\cdots,c_n$,

$$\|c_1C_{\psi_1}+\cdots+c_nC_{\psi_n}\|_e^2 \geq \sum_{\mathbf{d}\in\mathcal{D}_{k-1}(\alpha)} \left| \sum_{\substack{j\in\mathcal{M}_k(\alpha)\\ D_{k-1}(\psi_j,\alpha)=\mathbf{d}}} c_j \right|^2 \frac{1}{|d_1|}, \tag{9}$$

where $\mathbf{d}=(d_0,d_1,\cdots,d_{k-1})$.

For the case $k=2$, we need a calculation which appears in the proof of a more delicate version of the inequality (9); see Lemma 5.9 in [14]. Given ψ as above, with α in $F(\psi)$, convert it into a self-map u of the upper half-plane fixing the origin by $u=\tau_{\psi(\alpha)}\circ\psi\circ\tau_\alpha^{-1}$. Given our finite collection $\psi_1\cdots,\psi_n$, associate u_j to ψ_j in this manner. For $D>0$ we write $\Gamma_{\alpha,D}$ for the locus of the equation $(1-|z|^2)/(|\alpha-z|^2)=4D$, a circle internally tangent to $\partial\mathbb{D}$ at α. We have

$$\lim_{\Gamma_{\alpha,D}} \left\| (\overline{c_1}C_{\psi_1}^*+\cdots+\overline{c_n}C_{\psi_n}^*)\frac{k_z}{\|k_z\|}\right\|^2 = \sum_{\mathbf{d}\in\mathcal{D}_1(\alpha)} \left\| \sum_{\substack{j\in\mathcal{M}_2(\alpha)\\ D_1(\psi_j,\alpha)=\mathbf{d}}} \overline{c_j}k_{w_j}^+ \right\|_{H_+^2}^2 \tag{10}$$

where k_z is the Szego kernel for the Hardy space H^2 in the disk, H_+^2 is the Hardy space of the right half-plane Ω, $k_w^+(z)=1/(\overline{w}+z)$ is its reproducing kernel, $w_j=u_j'(0)/2-iDu_j''(0)$, and the limit is taken as $z\to\alpha$ along $\Gamma_{\alpha,D}$. Since $u_j''(0)$ necessarily has non-negative imaginary part, w_j is a complex number automatically lying in Ω. For further discussion of this circle of ideas, see [14]. We note for future

reference that a non-automorphism linear-fractional self-map ψ of $\mathbb{D}$ has order of contact two at the unique point in $F(\psi)$.

Finally, we need a variant of a result of Berkson [4] and Shapiro and Sundberg [19], which states that if $\psi_1, \cdots, \psi_n$ are distinct analytic self-maps of $\mathbb{D}$ and $J(\psi_i) = \{e^{i\theta} : |\psi_i(e^{i\theta})| = 1\}$, then for any complex constants $c_1, \cdots, c_n$,

$$\|c_1 C_{\psi_1} + \cdots + c_n C_{\psi_n}\|_e^2 \geq \frac{1}{2\pi} \sum_{j=1}^{n} |c_j|^2 |J(\psi_j)|, \tag{11}$$

where $|J|$ denotes the Lebesgue measure of J; see Exercise 9.3.2 in [8].

Theorem 1. *Let ψ be an analytic self-map of $\mathbb{D}$ such that C_ψ lies in $C^*(C_\varphi, \mathcal{K})$, where φ is as in (1). If $F(\psi)$ is empty, then C_ψ is compact.*

Proof. Suppose that C_ψ lies in $C^*(C_\varphi, \mathcal{K})$ and $F(\psi)$ is empty. We want to show that C_ψ is compact, or equivalently, that the matrix function

$$\Psi(C_\psi) = \begin{bmatrix} f_2 & f_3 \\ f_4 & f_1 \end{bmatrix}$$

is identically zero on $[0, s]$. Given a small $\epsilon > 0$ (size to be specified later), there exists B in $\mathcal{P}$ given by Equation (2) and equivalently by Equation (5), such that $\|C_\psi - B\| < \epsilon$. If we write

$$Y_1 = f(C_\varphi^* C_\varphi), Y_2 = g(C_\varphi C_\varphi^*), Y_3 = C_\varphi p(C_\varphi^* C_\varphi), Y_4 = C_\varphi^* q(C_\varphi C_\varphi^*),$$

and $Y = Y_1 + Y_2 + Y_3 + Y_4$, it is clear that $A_k \equiv Y_k \pmod{\mathcal{K}}$ for each i, and $\Psi(Y) = \Psi(A)$, where $A = A_1 + A_2 + A_3 + A_4$. Now using the representation (5) for B, we have

$$\|C_\psi - cC_z - A\|_e < \epsilon.$$

Since A is a finite linear combination of composition operators, we see from the inequality (11) that

$$\epsilon^2 > \|C_\psi - cC_z - A\|_e^2 \geq \frac{|J(\psi)|}{2\pi} + |c|^2.$$

From this we find that $|c| < \epsilon$, and since $\epsilon > 0$ is arbitrary, $|J(\psi)| = 0$. In particular we have $\|C_\psi - A\|_e < 2\epsilon$, hence

$$\left\| \begin{bmatrix} f_2 - g & f_3 - rp \\ f_4 - rq & f_1 - f \end{bmatrix} \right\| = \|\Psi(C_\psi - Y)\| = \|C_\psi - A\|_e < 2\epsilon.$$

It follows that $|f_3(t) - \sqrt{t}p(t)| < 2\epsilon$ for $0 \leq t \leq s$, and similarly for the other three matrix entries.

We will show that f_3 vanishes identically on $[0, s]$. Suppose not, so that its supremum norm $\|f_3\|_\infty$ is positive. Without loss of generality we may assume $8\epsilon < \|f_3\|_\infty$. It follows that there is a non-degenerate closed subinterval I of $[0, s]$, depending only on f_3 and not containing zero, with $\sqrt{t}|p(t)| \geq \|f_3\|_\infty/2$ for t in I. Thus

$$\int_I |p(t)|^2 dt \geq \frac{\|f_3\|_\infty^2 |I|}{4s}. \tag{12}$$

We return to this inequality below.

Now we want to apply Equation (10) to the linear combination A, which we write as

$$A = c_1 C_{\psi_1} + \cdots c_m C_{\psi_m}. \tag{13}$$

Recall that the normalized Szego kernel functions $k_z/\|k_z\|$ tend to zero weakly as $|z| \to 1$, and so

$$\|T\|_e = \|T^*\|_e \geq \limsup_{|z|\to 1} \left\| T^* \left(\frac{k_z}{\|k_z\|} \right) \right\|$$

for any bounded operator T on H^2. The linear-fractional maps $\psi_1, \cdots, \psi_m$ in Equation (13) are taken from the four lists in (6). The maps in each of these lists have a common angular derivative set (a singleton) and a single common first order data vector. For example, the maps ψ_i from the first list all have $F(\psi_i) = \{\eta\}$ and first order data vector $D_1(\psi_i, \eta) = (\eta, 1)$, which we call $\mathbf{d}_1$. The following table summarizes the corresponding information for each of the four lists:

TABLE I

ψ_i chosen from	$F(\psi_i)$	unique first-order data vector
$\{(\varphi \circ \sigma)_{n_1} :\ n_1 \geq 1\}$	$\{\eta\}$	$\mathbf{d}_1 = (\eta, 1)$
$\{(\sigma \circ \varphi)_{n_2} :\ n_2 \geq 1\}$	$\{\zeta\}$	$\mathbf{d}_2 = (\zeta, 1)$
$\{(\varphi \circ \sigma)_{n_3} \circ \varphi :\ n_3 \geq 0\}$	$\{\zeta\}$	$\mathbf{d}_3 = (\eta, \varphi'(\zeta))$
$\{(\sigma \circ \varphi)_{n_4} \circ \sigma :\ n_4 \geq 0\}$	$\{\eta\}$	$\mathbf{d}_4 = (\zeta, \sigma'(\eta))$

Since $C_\psi^*(k_z) = k_{\psi(z)}$, our hypothesis that $F(\psi)$ is empty says exactly that

$$\lim_{|z|\to 1} \left\| C_\psi^* \frac{k_z}{\|k_z\|} \right\|^2 = \lim_{|z|\to 1} \frac{1-|z|^2}{1-|\psi(z)|^2} = 0$$

(see [8], p.132) and thus

$$\begin{aligned} 4\epsilon^2 > \|C_\psi - A\|_e^2 \quad &\geq \quad \limsup_{|z|\to 1} \left\| (C_\psi^* - A^*) \frac{k_z}{\|k_z\|} \right\|^2 \\ &\geq \quad \lim_{\Gamma_{\zeta,D}} \left\| (\overline{c_1} C_{\psi_1}^* + \cdots + \overline{c_m} C_{\psi_m}^*) \frac{k_z}{\|k_z\|} \right\|^2 . \end{aligned}$$

We evaluate the limit on the right via Equation (10) with $\alpha = \zeta$. Note that $\mathcal{D}_1(\zeta) = \{\mathbf{d}_2, \mathbf{d}_3\}$. Discarding the (necessarily non-negative) $\mathbf{d}_2$ term from the right-hand side of Equation (10) yields

$$(14) \qquad 4\epsilon^2 > \lim_{\Gamma_{\zeta,D}} \left\| (\overline{c_1} C_{\psi_1}^* + \cdots + \overline{c_m} C_{\psi_m}^*) \frac{k_z}{\|k_z\|} \right\|^2 \geq \left\| \sum_{\substack{\zeta \in F(\psi_i) \\ D_1(\psi_i,\zeta) = \mathbf{d}_3}} \overline{c_i} k_{w_i}^+ \right\|_{H_+^2}^2 .$$

We can relabel $\psi_1, \cdots, \psi_m$ so that the relevant $\psi_i's$ occur at the beginning, starting with $i = 0$. Then the right-hand side of (14) becomes

$$(15) \qquad \begin{aligned} \left\| \sum_{k=0}^n \overline{c_k} k_{w_k}^+ \right\|_{H_+^2}^2 = \sum_{i,j=0}^n \overline{c_i} c_j \frac{1}{\overline{w_i} + w_j} \quad &= \quad \sum_{i,j=0}^n \overline{c_i} c_j \int_0^1 t^{\overline{w_i} + w_j - 1} dt \\ &= \quad \int_0^1 \left| \sum_{k=0}^n c_k t^{w_k} \right|^2 t^{-1} dt \end{aligned}$$

for appropriate $n \leq m-1$.

Let us evaluate c_k in terms of the polynomial p occuring in the upper right entry of the matrix function $\Psi(A) = \Psi(Y)$. If

$$p(z) = \sum_{k=0}^{n} b_k z^k,$$

we have

$$\begin{aligned} Y_3 = C_\varphi p(C_\varphi^* C_\varphi) &\equiv C_\varphi p(sC_\sigma C_\varphi)(\text{mod } \mathcal{K}) \\ &= C_\varphi p(sC_{\varphi\circ\sigma}) \\ &= \sum_{k=0}^{n} b_k s^k C_{(\varphi\circ\sigma)_k\circ\varphi}, \\ &= A_3, \end{aligned}$$

so that, relabeling if necessary, $\psi_k = (\varphi\circ\sigma)_k \circ \varphi$ and $c_k = b_k s^k$ for $k = 0, 1, \cdots, n$.

Next we compute w_k for $k = 0, 1, \cdots, n$. Let us convert ψ_k into a self-map U_k of the upper half-plane fixing the origin as described prior to Equation (10): $U_k = \tau_\eta \circ \psi_k \circ \tau_\zeta^{-1}$. We can do the same for the composition factors of $\psi_k = (\varphi\circ\sigma)_k \circ \varphi$. The map $\varphi\circ\sigma$ is a positive parabolic non-automorphism with fixed point η. Let $a > 0$ be its translation number, so that $\varphi\circ\sigma = \rho_a$. Thus for $k \geq 1$, $(\varphi\circ\sigma)_k = \rho_{ka}$, and its half-plane transplant $u_k = \tau_\eta \circ \rho_{ka} \circ \tau_\eta^{-1}$ satisfies $u_k''(0) = 2ika$ by Equation (7). We write v for the half-plane version of $\varphi : v = \tau_\eta \circ \varphi \circ \tau_\zeta^{-1}$. We have

$$U_k = \tau_\eta \circ \psi_k \circ \tau_\zeta^{-1} = (\tau_\eta \circ \rho_{ka} \circ \tau_\eta^{-1}) \circ (\tau_\eta \circ \varphi \circ \tau_\zeta^{-1}) = u_k \circ v.$$

Now $u_k'(0) = \rho_{ka}'(\eta) = 1$ and $v'(0) = |\varphi'(\zeta)| = \frac{1}{s}$, and we find

$$U_k'(0) = \frac{1}{s} \text{ and } U_k''(0) = v''(0) + \frac{2ika}{s^2}.$$

From the discussion following Equation (10) we see that

$$w_k = \frac{1}{2s} - iDv''(0) + k\left(\frac{2aD}{s^2}\right).$$

To this point D has been an arbitrary positive number. Let us choose D so that $2aD/s^2 = 1$ and put $\mu = \frac{1}{2s} - iDv''(0)$, a complex number with positive real part. Thus $w_k = \mu + k$ and we can write the right hand side of Equation (15) as

$$\begin{aligned} \int_0^1 \left| \sum_{k=0}^{n} b_k s^k t^k \right|^2 t^{2\text{Re}\mu - 1} dt &= \int_0^1 |p(st)|^2 t^{2\text{Re}\mu-1} dt \\ &= \frac{1}{s^{2\text{Re}\mu}} \int_0^s |p(t)|^2 t^{2\text{Re}\mu-1} dt \end{aligned}$$

We consider two cases: if $2\text{Re}\mu - 1 \geq 0$ then this last integral is at least

$$\frac{t_0^{2\text{Re}\mu-1}}{s^{2\text{Re}\mu}} \int_I |p(t)|^2 dt$$

where $t_0 > 0$ is the left-hand endpoint of I, and if $2\text{Re}\mu - 1 < 0$ it is at least

$$\frac{1}{s} \int_I |p(t)|^2 dt.$$

For small enough $\epsilon > 0$, either case of this inequality, combined with the inequality (14) and Equation (15), is incompatible with the inequality (12), yielding the desired contradiction. It follows that $f_3 \equiv 0$ on $[0, s]$. Entirely similar arguments show that f_1, f_2 and f_4 vanish identically on $[0, s]$, hence $\Psi(C_\psi) = 0$. $\square$

With Theorem 1 in hand, we can present our necessary conditions for membership in $C^*(C_\varphi, \mathcal{K})$.

Theorem 2. *Let φ be as in (1). Suppose ψ is an analytic self-map of $\mathbb{D}$ with C_ψ lying in $C^*(C_\varphi, \mathcal{K})$ and C_ψ not compact. Then either $\psi(z) = z$ or one of the following holds:*

(a) *$F(\psi) = \{\zeta\}, \psi(\zeta) = \eta$ and $\psi'(\zeta) = \varphi'(\zeta)$.*
(b) *$F(\psi) = \{\zeta\}, \psi(\zeta) = \zeta$ and $\psi'(\zeta) = 1$.*
(c) *$F(\psi) = \{\eta\}, \psi(\eta) = \zeta$ and $\psi'(\eta) = 1/\varphi'(\zeta)$.*
(d) *$F(\psi) = \{\eta\}, \psi(\eta) = \eta$ and $\psi'(\eta) = 1$.*
(e) *$F(\psi) = \{\zeta, \eta\}$ with $\psi(\zeta) = \eta, \psi'(\zeta) = \varphi'(\zeta), \psi(\eta) = \eta$ and $\psi'(\eta) = 1$.*
(f) *$F(\psi) = \{\zeta, \eta\}$ with $\psi(\eta) = \zeta, \psi'(\eta) = 1/\varphi'(\zeta), \psi(\zeta) = \zeta$ and $\psi'(\zeta) = 1$.*

Proof. If ψ has no finite angular derivative, then Theorem 1 guarantees that C_ψ is compact. Thus we may assume $F(\psi)$ is non-empty. We also assume ψ is not the identity, else there is nothing to prove. If C_ψ is in $C^*(C_\varphi, \mathcal{K})$, the density of the polynomial subalgebra $\mathcal{P}$ says that given ϵ, we may find a scalar c and a finite linear combination A of composition operators with associated maps from the lists (6) so that

$$\|C_\psi - A - cC_z\|_e < \epsilon.$$

As in the beginning of the proof of Theorem 1, we may then conclude that $|\psi(e^{i\theta})| < 1$ a.e., $|c| < \epsilon$, and that

$$\|C_\psi - A\| < 2\epsilon. \tag{16}$$

The self-maps of $\mathbb{D}$ which define the composition operators in the linear combination A appear among those in Table I above, along with their angular derivative sets (all singletons) and first order data vectors. Suppose that λ is in $F(\psi)$ and $D_1(\psi, \lambda) = \mathbf{d}$. If λ is not in $\{\zeta, \eta\}$, then the inequality (8) gives

$$\|C_\psi - A\|_e^2 \geq \frac{1}{|\psi'(\lambda)|}, \tag{17}$$

contradicting (16). Similarly, if $\lambda = \zeta$ and $\mathbf{d}$ is neither $\mathbf{d_2}$ nor $\mathbf{d_3}$ from Table I, or if $\lambda = \eta$ and $\mathbf{d}$ is neither $\mathbf{d_1}$ nor $\mathbf{d_4}$, the inequality (8) and Table I again imply (17). It follows that if $F(\psi)$ is a singleton, one of conditions (a)-(d) must hold.

The remainder of the proof considers the possibility that $F(\psi) = \{\zeta, \eta\}$. The Julia-Caratheodory theory says a non-identity analytic self-map of $\mathbb{D}$ cannot have fixed points at distinct points ζ, η in $\partial\mathbb{D}$ with derivative 1 at each. If we have both $\psi(\zeta) = \eta, \psi'(\zeta) = \varphi'(\zeta)$ and $\psi(\eta) = \zeta, \psi'(\eta) = 1/\varphi'(\zeta)$, then $\psi \circ \psi$ fixes both ζ and η with derivative 1 at each, so that $\psi \circ \psi$ is the identity map, contradicting the observation above that $|\psi(e^{i\theta})| < 1$ almost everywhere. Thus if $F(\psi) = \{\zeta, \eta\}$, either (e) or (f) must hold, completing the proof. $\square$

We will see in Section 5 that there are indeed maps ψ of types (e) and (f) in Theorem 2 for which C_ψ belongs to $C^*(C_\varphi, \mathcal{K})$.

3. The C^*-algebra induced by parabolic non-automorphisms

Let us write $\mathcal{B}(H^2)$ for the algebra of bounded operators on H^2. A bounded operator T on H^2 is essentially normal if T and T^* commute modulo $\mathcal{K}$; normal operators and compact operators give trivial examples of essentially normal operators. The only normal composition operators C_ψ are those of the form $\psi(z) = az, |a| \leq 1$. Bourdon, Levi, Narayan, and Shapiro [1] showed that if ψ is linear-fractional with $\|\psi\|_\infty = 1$ and not a rotation, then C_ψ is essentially normal exactly when ψ is a parabolic non-automorphism. Let us select a point γ in $\partial\mathbb{D}$ and consider the set $\{\rho_a : \text{Re } a > 0\}$ of all parabolic non-automorphisms fixing γ. Here, as earlier, a is the translation number for ρ_a. Any two of the maps ρ_a commute under composition and in fact $\rho_a \circ \rho_b = \rho_{a+b}$, so C_{ρ_a} and C_{ρ_b} commute. One can easily check that the Krein adjoint of ρ_a is $\rho_{\overline{a}}$. Since $\rho_a'(\gamma) = 1$, it follows from Equation (4) that $C_{\rho_a}^* = C_{\rho_{\overline{a}}} + K$ for some compact operator K. A recent theorem of Montes-Rodríguez, Ponce-Escudero and Shkarin [18] shows that C_{ρ_a} is irreducible. Moreover $C^*(C_{\rho_a})$, the unital C^*-algebra generated by C_{ρ_a}, contains the commutator of C_{ρ_a} and $C_{\rho_a}^*$ which we know is compact but non-zero. Thus $C^*(C_{\rho_a})$ contains $\mathcal{K}$ and $C^*(C_{\rho_a})/\mathcal{K}$ is commutative. Now let $\mathbb{P}_\gamma$ denote the set of all composition operators C_ρ, where ρ, fixing γ, ranges over $\{\rho_a : \text{Re } a > 0\}$. We write $C^*(\mathbb{P}_\gamma)$ for the unital C^*-algebra generated by the operators in $\mathbb{P}_\gamma$. Clearly $C^*(\mathbb{P}_\gamma)$ contains $\mathcal{K}$, and, by the above remarks, $C^*(\mathbb{P}_\gamma)/\mathcal{K}$ is also commutative. In this section we compute and apply the Gelfand representation of this quotient algebra.

We begin with two lemmas.

Lemma 1. *For $a > 0$ there is an operator $A \geq 0$ and a compact operator K with $C_{\rho_a} = A + K$.*

Proof. Then

$$\begin{aligned} C_{\rho_{a/2}} &= \frac{1}{2}(C_{\rho_{a/2}} + C_{\rho_{a/2}}^*) + \frac{1}{2}(C_{\rho_{a/2}} - C_{\rho_{a/2}}^*) \\ &\equiv B + J \end{aligned}$$

where B is self-adjoint and J is compact. Thus $C_{\rho_a} = C_{\rho_{a/2}} C_{\rho_{a/2}} = (B+J)^2 = B^2 + (BJ + JB + J^2)$. Since B^2 is positive and $BJ + JB + J^2$ is compact, we are done. □

Lemma 2. *Let a, b be positive with $b/a = m/n$, with m and n positive integers. Suppose $0 < \lambda \leq 1$ and there is a sequence f_k of unit vectors in H^2 converging weakly to zero such that*

$$\|(C_{\rho_a} - \lambda) f_k\| \to 0.$$

Then

$$\|(C_{\rho_b} - \lambda^{m/n}) f_k\| \to 0.$$

Proof. First observe that

$$C_{\rho_a}^m - \lambda^m = [C_{\rho_a}^{m-1} + \lambda C_{\rho_a}^{m-2} + \cdots + \lambda^{m-2} C_{\rho_a} + \lambda^{m-1}][C_{\rho_a} - \lambda].$$

In particular, $\|(C_{\rho_a}^m - \lambda^m) f_k\| \to 0$ as $k \to \infty$. Since $C_{\rho_a}^m = C_{\rho_b}^n$,

$$\|(C_{\rho_b}^n - \lambda^m) f_k\| \to 0.$$

Also note that we may factor $C_{\rho_b}^n - \lambda^m = C_{\rho_b}^n - (\lambda^{m/n})^n$ as

$$[C_{\rho_b}^{n-1} + \lambda^{m/n} C_{\rho_b}^{n-2} + \cdots + (\lambda^{m/n})^{n-2} C_{\rho_b} + (\lambda^{m/n})^{n-1}][C_{\rho_b} - \lambda^{m/n}].$$

Apply Lemma 1 to C_{ρ_b} to write

$$C_{\rho_b}^{n-1} + \lambda^{m/n} C_{\rho_b}^{n-2} + \cdots + (\lambda^{m/n})^{n-2} C_{\rho_b} + (\lambda^{m/n})^{n-1} I = T + (\lambda^{m/n})^{n-1} I + K$$

for some positive T and compact K. We have

$$\|(C_{\rho_b}^n - \lambda^m) f_k\| = \|(T + (\lambda^{m/n})^{n-1} + K)(C_{\rho_b} - \lambda^{m/n}) f_k\|. \tag{18}$$

Since K is compact, $\|K(C_{\rho_b} - \lambda^{m/n}) f_k\| \to 0$ as $k \to \infty$. Since the left-hand side of Equation (18) goes to 0 as $k \to \infty$, we see, writing $c = (\lambda^{m/n})^{n-1}$ that

$$\|(T + cI)(C_{\rho_b} - \lambda^{m/n}) f_k\| \to 0.$$

But $\|(T + cI)(C_{\rho_b} - \lambda^{m/n}) f_k\|^2$ is equal to

$$\begin{aligned} \|T(C_{\rho_b} - \lambda^{m/n}) f_k\|^2 \quad &+ \quad 2c\langle T(C_{\rho_b} - \lambda^{m/n}) f_k, (C_{\rho_b} - \lambda^{m/n}) f_k \rangle \\ &+ \quad c^2 \|(C_{\rho_b} - \lambda^{m/n}) f_k\|^2 \geq c^2 \|(C_{\rho_b} - \lambda^{m/n}) f_k\|^2 \end{aligned}$$

where the last inequality uses the positivity of T, so that $\|(C_{\rho_b} - \lambda^{m/n}) f_k\| \to 0$. $\square$

The essential spectrum $\sigma_e(T)$ of a bounded operator on H^2 is by definition the spectrum of the coset $[T]$ in $\mathcal{B}(H^2)/\mathcal{K}$. We recall from [15] that if $a > 0, \sigma_e(C_{\rho_a}) = [0, 1]$. We will need the notion of joint essential spectrum, which is treated by Dash in [9]. If Re $a > 0$, the coset $[C_{\rho_a}]$ of C_{ρ_a} modulo $\mathcal{K}$ will also be denoted by x_a. By either Lemma 1 or the discussion preceeding it, x_a is self-adjoint. Given a and b, the joint essential spectrum $\sigma_e(C_{\rho_a}, C_{\rho_b})$ is defined to be the joint spectrum $\sigma(x_a, x_b)$ of the pair x_a, x_b in the Calkin algebra $\mathcal{B}(H^2)/\mathcal{K}$. This set coincides with the joint spectrum in the commutative unital subalgebra $C^*(x_a, x_b)$ generated by x_a and x_b. If $\mathcal{M}$ is the maximal ideal space of this algebra, and $\widehat{\ }$ denotes the Gelfand transform, then the map $\ell \mapsto (\widehat{x_a}(\ell), \widehat{x_b}(\ell))$ is a homeomorphism of $\mathcal{M}$ onto $\sigma(x_a, x_b)$. Let us assume that a and b are positive. A theorem of Dash [9] states, in this context, using $C_{\rho_a}^* \equiv C_{\rho_a} (\text{mod } \mathcal{K})$ and similarly for C_{ρ_b}, that (λ, μ) lies in $\sigma_e(C_{\rho_a}, C_{\rho_b})$ if and only if there exists a sequence $\{f_k\}$ of unit vectors in H^2, converging weakly to zero, such that $\|(C_{\rho_a} - \lambda) f_k\|$ and $\|(C_{\rho_b} - \mu) f_k\|$ both tend to zero as $k \to \infty$.

Corollary 1. *Suppose that a, b are positive and b/a is rational. Then*

$$\sigma_e(C_{\rho_a}, C_{\rho_b}) = \{(t^a, t^b) : 0 \leq t \leq 1\}.$$

Proof. We know that

$$\sigma_e(C_{\rho_a}, C_{\rho_b}) \subset \sigma_e(C_{\rho_a}) \times \sigma_e(C_{\rho_b}) = [0, 1] \times [0, 1].$$

Let $0 < \lambda \leq 1$. Since $\lambda \in \sigma_e(C_{\rho_a})$, we may find unit vectors f_k with $f_k \to 0$ weakly and $\|(C_{\rho_a} - \lambda) f_k\| \to 0$. By Lemma 2, $\|(C_{\rho_b} - \lambda^{b/a}) f_k\| \to 0$ as well, so that $(\lambda, \lambda^{b/a})$ is in $\sigma_e(C_{\rho_a}, C_{\rho_b})$ by Dash's theorem. Setting $t^a = \lambda$ we have $\{(t^a, t^b) : 0 < t \leq 1\} \subset \sigma_e(C_{\rho_a}, C_{\rho_b})$. The set on the right is compact in $\mathbb{R}^2$, so it contains $(0, 0)$ as well, giving $\{(t^a, t^b) : 0 \leq t \leq 1\} \subset \sigma_e(C_{\rho_a}, C_{\rho_b})$. Conversely, if $(\lambda, \mu) \in \sigma_e(C_{\rho_a}, C_{\rho_b})$, Dash's theorem gives the existence of a sequence of unit vectors f_k converging weakly to 0 with both $\|(C_{\rho_a} - \lambda) f_k\|$ and $\|(C_{\rho_b} - \mu) f_k\|$ tending to zero as $k \to \infty$. If $\lambda > 0$, $\|(C_{\rho_b} - \lambda^{b/a}) f_k\| \to 0$ by Lemma 2. Thus $\mu = \lambda^{b/a}$ and $(\lambda, \mu) = (\lambda, \lambda^{b/a}) = (t^a, t^b)$ for some t, $0 \leq t \leq 1$. If $\mu > 0$, the symmetric result says $\lambda = \mu^{a/b} > 0$, and, putting $\mu = t^b$, $(\lambda, \mu) = (\mu^{a/b}, \mu) = (t^a, t^b)$, again of the desired form. As for $(0, 0)$, we already know it lies in $\sigma_e(C_{\rho_a}, C_{\rho_b})$, and of course it has the form $(0^a, 0^b)$. $\square$

We will need the fact that on the domain $\Omega = \{a : \text{Re } a > 0\}$ the map $a \mapsto C_{\rho_a}$ is a holomorphic function of a in the operator norm topology; see for example the discussion in the proof of Theorem 6.1 in [6]. We continue to denote the coset of C_{ρ_a} by x_a and to keep in mind that when $a > 0$, $\sigma(x_a) = [0, 1]$.

Theorem 3. *There is a unique $*$-isomorphism $\Gamma : C([0, 1]) \to C^*(\mathbb{P}_\gamma)/\mathcal{K}$ such that $\Gamma(t^a) = [C_{\rho_a}]$ for all $a \in \Omega$.*

Proof. First consider $a = 1$ and $x_1 = [C_{\rho_1}]$. Since $\sigma(x_1) = [0, 1]$ we may define a $*$-isomorphism $\Gamma : C([0, 1]) \mapsto C^*(x_1)$ by sending p to $p(x_1)$ for any polynomial p. Fix any rational number $r > 0$. By Corollary 1,

$$\sigma(x_1, x_r) = \sigma_e(C_{\rho_1}, C_{\rho_r}) = \{(t, t^r) : 0 \le t \le 1\}.$$

The map $p(x_1, x_r) \mapsto p(z_1, z_2)$, where p is a two-variable polynomial, extends to a unique $*$-isomorphism of $C^*(x_1, x_r)$ onto $C(\sigma(x_1, x_r))$. Since $\sigma(x_1, x_r)$ is homeomorphic to $[0, 1]$ via the map $t \mapsto (t, t^r)$, we see that $p(x_1, x_r) \mapsto p(t, t^r)$ defines a $*$-isomorphism of $C^*(x_1, x_r)$ onto $C([0, 1])$. Let $\tilde{\Gamma}$ denote the inverse of this map, that is $\tilde{\Gamma} : p(t, t^r) \mapsto p(x_1, x_r)$. Since polynomials in t span $C([0, 1])$, x_1 generates the C^*-algebra $C^*(x_1, x_r)$ and $C^*(x_1) = C^*(x_1, x_r)$. It follows that $\tilde{\Gamma} = \Gamma$. Since r is arbitrary in the set $\mathbb{Q}_+$ of positive rationals, we have shown that

$$C^*(\{x_r : r \in \mathbb{Q}_+\}) = C^*(x_1).$$

Moreover, $\Gamma(t^r) = x_r, r \in \mathbb{Q}_+$. It is easy to see that the map $a \mapsto t^a$ is a norm-holomorphic map of the right half plane into $C([0, 1])$ and thus that $a \mapsto \Gamma(t^a)$ is norm- holomorphic from the right half-lane to $\mathcal{B}(H^2)/\mathcal{K}$, as is the function $a \mapsto x_a$. We have seen that these functions agree on $\mathbb{Q}_+$, hence they must agree on the right half-plane Ω. □

We record three immediate consequences.

Corollary 2. *If $a_1, \cdots, a_n$ lie in the right half-plane Ω, then*

$$\sigma_e(C_{\rho_{a_1}}, \cdots, C_{\rho_{a_n}}) = \{(t^{a_1}, \cdots, t^{a_n}) : 0 \le t \le 1\}.$$

Corollary 3. *If ρ is a parabolic non-automorphism fixing γ, then $C^*(C_\rho) = C^*(\mathbb{P}_\gamma)$.*

Corollary 4. *If φ is as in (1), then $\mathbb{P}_\zeta$ and $\mathbb{P}_\eta$ are both subsets of $C^*(C_\varphi, \mathcal{K})$.*

4. Linear-fractional maps

The goal of this section is to find all linear-fractional ψ with C_ψ in $C^*(C_\varphi, \mathcal{K})$, where φ satisfies the conditions of (1). Since C_ψ is compact if $\|\psi\|_\infty < 1$, our interest is in the case $\|\psi\|_\infty = 1$.

Lemma 3. *If φ is as in (1), $C^*(C_\varphi, \mathcal{K})$ contains C_ψ for all linear-fractional $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(\zeta) = \eta$, $\psi'(\zeta) = \varphi'(\zeta)$ and $\psi(\mathbb{D})$ properly contained in $\varphi(\mathbb{D})$.*

Proof. Set $\tau = \varphi^{-1} \circ \psi$, noting that the hypothesis $\psi(D) \subset \varphi(\mathbb{D})$ means that τ is well-defined. Since this containment is proper, and $\tau'(\zeta) = 1$, τ is a parabolic non-automorphism with fixed point ζ. Since $\varphi \circ \tau = \psi$, $C_\psi = C_\tau C_\varphi$. By Corollary 4, $C_\tau \in C^*(C_\varphi, \mathcal{K})$, from which the conclusion follows. □

Now consider the parabolic non-automorphism $\rho = \varphi \circ \sigma$. The unique fixed point for ρ and its iterates $(\rho)_n$ is η. Fix an integer $n \geq 1$ and let $\varphi_1 = (\rho)_n \circ \varphi$. Clearly $\varphi_1(\mathbb{D})$ is properly contained in $\varphi(\mathbb{D})$. Note that $\varphi_1(\zeta) = \eta$ and, since $\rho'(\eta) = 1$, $\varphi_1'(\zeta) = \varphi'(\zeta)$. It follows from Lemma 3 that $C^*(C_{\varphi_1}, \mathcal{K})$ is contained in $C^*(C_\varphi, \mathcal{K})$. Let i denote the corresponding inclusion map. Since $i(\mathcal{K}) = \mathcal{K}$, i induces a $*$-homomorphism

$$\hat{i} : C^*(C_{\varphi_1}, \mathcal{K})/\mathcal{K} \to C^*(C_\varphi, \mathcal{K})/\mathcal{K}$$

given by $\hat{i}([T]) = [T]$, where $[T]$ denotes the coset, modulo $\mathcal{K}$ of the operator T. Note that $\hat{i}$ is itself an inclusion. Also observe that the map $\Psi : C^*(C_\varphi, \mathcal{K}) \to \mathcal{D}$ induces a $*$-isomorphism $\Phi : C^*(C_\varphi, \mathcal{K})/\mathcal{K} \to \mathcal{D}$ given by $\Phi([T]) = \Psi(T)$. Let Φ_1 denote the corresponding $*$-isomorphism $\Phi_1 : C^*(C_{\varphi_1}, \mathcal{K})/\mathcal{K} \to \mathcal{D}$. Keep in mind that Φ_1 should be defined by $\Phi_1([T]) = \Psi_1(T)$, where $\Psi_1 : C^*(C_{\varphi_1}, \mathcal{K}) \to \mathcal{D}$ is associated to φ_1 as Ψ is associated to φ. Thus if B in $C^*(C_{\varphi_1}, \mathcal{K})$ is given by (2), *but with φ replaced by φ_1*, then $\Psi_1(B)$ is given by (3). We have a commutative diagram

$$\begin{tikzcd} C^*(C_{\varphi_1}, \mathcal{K})/\mathcal{K} \arrow[r, "\hat{i}"] \arrow[d, "\Phi_1"'] & C^*(C_\varphi, \mathcal{K})/\mathcal{K} \arrow[d, "\Phi"] \\ \mathcal{D} \arrow[r, "\Lambda"] & \mathcal{D} \end{tikzcd}$$

where $\Lambda = \Phi \circ \hat{i} \circ \Phi_1^{-1}$. We seek to identify Λ explicitly.

Lemma 4. *For any element F in $\mathcal{D}$,*

$$(\Lambda F)(t) = F(t^{2n+1}/s^{2n}),\ 0 \leq t \leq s. \tag{19}$$

Proof. For the purposes of the proof, we use Λ to denote the map given by formula (19), and then show, with this redefinition, that it coincides with $\Phi \circ \hat{i} \circ \Phi_1^{-1}$, that is, that $\Lambda \circ \Phi_1 = \Phi \circ \hat{i}$. Recall that $C_\varphi^* \equiv sC_\sigma \pmod{\mathcal{K}}$ so that

$$\begin{aligned} C_{\varphi_1} = C_\varphi C_{(\rho)_n} &= C_\varphi (C_\sigma C_\varphi)^n \\ &\equiv \frac{1}{s^n} C_\varphi (C_\varphi^* C_\varphi)^n \pmod{\mathcal{K}}, \end{aligned}$$

and, taking adjoints, $C_{\varphi_1}^* \equiv \frac{1}{s^n} C_\varphi^* (C_\varphi C_\varphi^*)^n$ modulo the compacts. Calculations using these two facts show that if we write $y = [C_{\varphi_1}]$ and $x = [C_\varphi]$, we have, for each non-negative integer m,

$$(y^*y)^m = \frac{1}{s^{2nm}} (x^*x)^{(2n+1)m},$$

$$(yy^*)^m = \frac{1}{s^{2nm}} (xx^*)^{(2n+1)m}.$$

$$y(y^*y)^m = \frac{1}{s^{(2m+1)n}} x(x^*x)^{(2n+1)m+n},$$

and

$$y^*(yy^*)^m = \frac{1}{s^{(2m+1)n}} x^*(xx^*)^{(2n+1)m+n}.$$

The left-hand sides in these four equations are elements in $C^*(C_{\varphi_1},\mathcal{K})/\mathcal{K}$, while the right-hand sides represent the same objects as elements of $C^*(C_\varphi,\mathcal{K})/\mathcal{K}$. We first act on $y(y^*y)^m$ by $\hat{i}$, followed by Φ. We then act on $y(y^*y)^m$ by Φ_1, followed by Λ (as defined by Equation (19)). As the reader can see from the following picture, we end up with a common result, the matrix function in the lower right-hand corner.

$$\begin{tikzcd}
y(y^*y)^m \arrow[r, "\hat{i}"] \arrow[d, "\Phi_1"'] & \frac{1}{s^{(2m+1)n}}x(x^*x)^{(2n+1)m+n} \arrow[d, "\Phi"] \\
\begin{bmatrix} 0 & \sqrt{t}\ t^m \\ 0 & 0 \end{bmatrix} \arrow[r, "\Lambda"] & \begin{bmatrix} 0 & \frac{\sqrt{t}\ t^{(2n+1)m+n}}{s^{(2m+1)n}} \\ 0 & 0 \end{bmatrix}
\end{tikzcd}$$

One can check that we also arrive at common values when $\Lambda\circ\Phi_1$ and $\Phi\circ\hat{i}$ act on $(y^*y)^m$, and similarly for $(yy^*)^m$ and $y^*(yy^*)^m$. Since elements of the form $(y^*y)^m$, $(yy^*)^m$, $y(y^*y)^m$ and $y^*(yy^*)^m$, together with the identity, span $C^*(C_{\varphi_1},\mathcal{K})/\mathcal{K}$, we have $\Lambda\circ\Phi_1 = \Phi\circ\hat{i}$ as desired □

It is clear from (19) that Λ is an automorphism of $\mathcal{D}$. It follows that $\hat{i}$ is an isomorphism and thus that i has range equal to all of $C^*(C_\varphi,\mathcal{K})$, that is,

$$C^*(C_{\varphi_1},\mathcal{K}) = C^*(C_\varphi,\mathcal{K}). \tag{20}$$

More generally, we have the following result.

Theorem 4. *Let ψ be a linear-fractional map of $\mathbb{D}$, not an automorphism, with $\psi(\zeta)=\varphi(\zeta)$ and $\psi'(\zeta)=\varphi'(\zeta)$, where φ is as in (1). Then $C^*(C_\psi,\mathcal{K}) = C^*(C_\varphi,\mathcal{K})$.*

Proof. The circles $\varphi(\partial\mathbb{D})$ and $\psi(\partial\mathbb{D})$ are both internally tangent to $\partial\mathbb{D}$ at η. If $\varphi(\mathbb{D})$ is a proper subset of $\psi(\mathbb{D})$, then

$$C^*(C_\varphi,\mathcal{K}) \subset C^*(C_\psi,\mathcal{K}) \tag{21}$$

by Lemma 3. Suppose on the other hand that $\psi(\mathbb{D})\subset\varphi(\mathbb{D})$. If a is the (necessarily positive) translation number of the parabolic map $\varphi\circ\sigma$ (so that $\varphi\circ\sigma=\rho_a$ in the terminology of Section 2), then $(\varphi\circ\sigma)_n=\rho_{na}$, and the radius of the disk $\rho_{na}(\mathbb{D})$ shrinks to zero as $n\to\infty$. Thus there exists n with $\rho_{na}(\mathbb{D})$ properly contained in $\psi(D)$. If $\varphi_1=(\varphi\circ\sigma)_n\circ\varphi=\rho_{na}\circ\varphi$, then $\varphi_1(\mathbb{D})$ is also properly contained in $\psi(\mathbb{D})$. Since $\varphi_1(\zeta)=\eta=\psi(\zeta)$ and $\varphi_1'(\zeta)=\rho_{na}'(\eta)\varphi'(\zeta)=\varphi'(\zeta)=\psi'(\zeta)$, Lemma 3 implies that $C^*(C_\psi,\mathcal{K})$ contains $C^*(C_{\varphi_1},\mathcal{K})$, which by (20) coincides with $C^*(C_\varphi,\mathcal{K})$. The result is that (21) holds, whatever the relationship between the disks $\varphi(\mathbb{D})$ and $\psi(\mathbb{D})$. The statement of the theorem is symmetric in φ and ψ, so symmetry implies that the containment reverse to that in (21) also holds, completing the proof. □

Theorem 5. *Suppose φ is as in (1). Let ψ, not the identity, be any linear-fractional self-map of $\mathbb{D}$ with $\|\psi\|_\infty=1$. Then C_ψ is in $C^*(C_\varphi,\mathcal{K})$ if and only if ψ is not an automorphism and one of the following conditions holds:*

(a) $\psi(\zeta)=\eta$ *and* $\psi'(\zeta)=\varphi'(\zeta)$.
(b) $\psi(\zeta)=\zeta$ *and* $\psi'(\zeta)=1$.
(c) $\psi(\eta)=\zeta$ *and* $\psi'(\eta)=1/\varphi'(\zeta)$.
(d) $\psi(\eta)=\eta$ *and* $\psi'(\eta)=1$.

Proof. The "only if" statement follows immediately from Theorem 2 and the hypothesis that ψ is linear-fractional. Conversely, let ψ be a linear-fractional map which is not an automorphism. If ψ is parabolic with fixed point at either ζ or η, the result follows from Corollary 4; this handles the cases (b) and (d). If ψ is as in (a), then we have $C_\psi \in C^*(C_\varphi, \mathcal{K})$ by Theorem 4. Finally, if ψ satisfies condition (c), then its Krein adjoint σ_ψ is a linear-fractional self-map of $\mathbb{D}$, not an automorphism, which satisfies condition (a), so that $C_{\sigma_\psi} \in C^*(C_\varphi, \mathcal{K})$. Since $C_\psi^* \equiv sC_{\sigma_\psi}$ modulo the compacts, this completes the argument. $\square$

The maps satisfying (a)-(d) in Theorem 5 can be described more explicitly. Given a point γ on $\partial\mathbb{D}$, let us write $\rho_{\gamma,a}$ for the unique parabolic map fixing γ with translation number a. This will be a self-map of $\mathbb{D}$ when Re $a \geq 0$, but when Re $a < 0$, $\rho_{\gamma,a}$ takes $\mathbb{D}$ onto a larger disk, whose boundary is externally tangent to $\partial\mathbb{D}$ at γ. Clearly, the linear-fractional non-automorphisms of $\mathbb{D}$ satisfying (b) or (d) are, respectively, $\rho_{\zeta,a}$ or $\rho_{\eta,a}$ with Re $a > 0$. Imaginary a gives an automorphism of $\mathbb{D}$, but in this case $C^*(C_\varphi, \mathcal{K})$ does not contain the corresponding composition operator.

Consider now a linear-fractional non-automorphism ψ of $\mathbb{D}$ satisfying (a). If $\psi(\mathbb{D}) \subset \varphi(\mathbb{D})$, we can define $\rho = \psi \circ \varphi^{-1}$ which fixes η and carries $\mathbb{D}$ to $\mathbb{D}$. Moreover, $\rho'(\eta) = \psi'(\zeta)/\varphi'(\zeta) = 1$, so ρ is parabolic; say $\rho = \rho_{\eta,a}$ where Re $a \geq 0$, and we find $\psi = \rho_{\eta,a} \circ \varphi$. On the other hand, if ψ satisfies (a) and $\varphi(\mathbb{D})$ is a proper subset of $\psi(\mathbb{D})$, we put $\rho = \varphi \circ \psi^{-1}$ which is again a parabolic self-map of $\mathbb{D}$; this time a non-automorphism. Thus $\rho = \rho_{\eta,a}$ with Re $a > 0$ and we find $\psi = \rho_{\eta,a}^{-1} \circ \varphi = \rho_{\eta,-a} \circ \varphi$. If b is the unique positive number with $\rho_{\eta,b}(\mathbb{D}) = \varphi(\mathbb{D})$, then $\rho_{\eta,-a} \circ \varphi$ is a non-automorphism self-map of $\mathbb{D}$ exactly when Re $a < b$. Rephrasing and summarizing, we conclude that the non-automorphisms ψ of $\mathbb{D}$ satisfying (a) are precisely the maps of the form $\rho_{\eta,a} \circ \varphi$, Re $a > -b$. Similarly, if c is the unique positive number with $\rho_{\zeta,c}(\mathbb{D}) = \sigma(\mathbb{D})$, then the non-automorphisms ψ satisfying (c) are exactly the maps of the form $\psi = \rho_{\zeta,a} \circ \sigma$ with Re $a > -c$. The next result shows that the positive translation numbers b and c are nicely related to each other, and to the translation numbers of the positive parabolic non-automorphisms $\varphi \circ \sigma$ and $\sigma \circ \varphi$.

Theorem 6. *Let b and c be the unique positive numbers with $\rho_{\eta,b}(\mathbb{D}) = \varphi(\mathbb{D})$ and $\rho_{\zeta,c}(\mathbb{D}) = \sigma(\mathbb{D})$, respectively. We have $c = |\varphi'(\zeta)|b$, and moreover, $\varphi \circ \sigma = \rho_{\eta,2b}$ and $\sigma \circ \varphi = \rho_{\zeta,2c}$.*

Proof. Clearly there is no loss of generality in assuming that $\zeta = 1$. The non-affine linear-fractional self-maps of $\mathbb{D}$ which send 1 to $\eta \in \partial\mathbb{D}$ can be written in the form

$$\varphi(z) = \eta\ \frac{(1 + s + sd)z + (d - s - sd)}{z + d}$$

where $s = |\varphi'(1)|$ and Re $\frac{d-1}{d+1} \geq s$ (see [3]). A computation shows that $\varphi'(1) = \eta s$ and $\varphi''(1) = -2\eta s/(1 + d)$. The image of the unit circle under φ is a circle with curvature $\kappa_1 = |\varphi'(1)|^{-1}$Re $[1 + \varphi''(1)/\varphi'(1)] = \frac{1}{s}$Re $[1 - \frac{2}{1+d}]$. Since

$$\sigma(z) = \frac{\overline{\eta}(1 + s + s\overline{d})z - 1}{-\overline{\eta}(\overline{d} - s - s\overline{d})z + \overline{d}},$$

we find that $\sigma'(\eta) = \overline{\eta}/s$ and

$$\sigma''(\eta) = \frac{2\overline{\eta}^2(\overline{d} - s - s\overline{d})}{s^2(1 + \overline{d})},$$

so that the image of the unit circle under σ is a circle with curvature

$$\kappa_2 = s\mathrm{Re}\,\left\{1 + \eta\,\frac{2\overline{\eta}^2(\overline{d} - s - s\overline{d})}{s^2(1+\overline{d})} \cdot \frac{s}{\overline{\eta}}\right\} = 2 - 2\mathrm{Re}\,\frac{1}{1+d} - s.$$

A positive parabolic non-automorphism fixing 1 and corresponding to translation by a has the form $((2+d)z-1)/(z+d)$ where $a = -2/(d+1)$; by the above calculations the image of the unit circle under this map has curvature $1 + \mathrm{Re}\ a$. Thus, if $\varphi(1) = \eta$ and $|\varphi'(1)| = s$, the unique positive value b such that the curvature of $\varphi(\partial\mathbb{D})$ is equal to the curvature of the image of $\partial\mathbb{D}$ under the positive parabolic map which corresponds to translation by $b > 0$ satisfies

$$1 + b = \frac{1}{s}\ \mathrm{Re}\ \left(1 - \frac{2}{1+d}\right);$$

that is,

$$b = \frac{1}{s}\ \mathrm{Re}\left(1 - \frac{2}{1+d}\right) - 1.$$

Similarly, the curvature of the circle $\sigma(\partial\mathbb{D})$ is equal to the curvature of the circle which is the image of the unit circle under the positive parabolic non-automorphism corresponding to translation by c precisely when $c = 1 - s - 2\mathrm{Re}\frac{1}{1+d}$. Thus $c = sb$. This conclusion also holds when φ is an affine map, $\varphi(z) = \eta(sz + 1 - s)$, where the computations are easier.

For the final statement, let $\psi = \rho_{\eta,-b} \circ \varphi$, so that ψ is an automorphism of $\mathbb{D}$ and $\varphi = \rho_{\eta,b} \circ \psi$. Since the Krein adjoint of an automorphism is its inverse, we have

$$\sigma = \sigma_\varphi = \sigma_\psi \circ \sigma_{\rho_{\eta,b}} = \psi^{-1} \circ \rho_{\eta,\overline{b}} = \psi^{-1} \circ \rho_{\eta,b}$$

and thus $\varphi \circ \sigma = \rho_{\eta,b} \circ \psi \circ \psi^{-1} \circ \rho_{\eta,b} = \rho_{\eta,2b}$. Similarly, $\sigma \circ \varphi = \rho_{1,2c} = \rho_{\zeta,2c}$. □

The remarks preceeding Theorem 6 express the linear-fractional maps ψ with C_ψ belonging to $C^*(C_\varphi, \mathcal{K})$ in terms of $\varphi, \sigma, \rho_{\eta,a}$ and $\rho_{\zeta,a}$ for appropriate ranges of the translation numbers a. We describe below a corresponding operator-theoretic description of C_ψ modulo $\mathcal{K}$, in terms of the polar factors of C_φ and C_φ^*. In [15] it was shown that every operator B in $C^*(C_\varphi, \mathcal{K})$ has a representation generalizing Equation (2) and having the form $B = T + K$ with K compact and

$$T = cI + f(C_\varphi^* C_\varphi) + g(C_\varphi C_\varphi^*) + Uh(C_\varphi^* C_\varphi) + U^* k(C_\varphi C_\varphi^*), \tag{22}$$

where f and h are continuous on $\sigma(C_\varphi^* C_\varphi)$, g and k are continuous on $\sigma(C_\varphi C_\varphi^*)$, all four functions vanish at zero, and U is the partial isometry polar factor (which in this case is unitary) of C_φ. The restrictions of f, g, h and k to the interval $[0, s]$, which coincides with both of the essential spectra $\sigma_e(C_\varphi^* C_\varphi)$ and $\sigma_e(C_\varphi C_\varphi^*)$, are uniquely determined by B. We call T a *distinguished representative* of the coset $[B]$, and recall from [15] that

$$\Psi(B) = \Psi(T) = \begin{bmatrix} c+g & h \\ k & c+f \end{bmatrix}.$$

We start with the operator $(C_\varphi^* C_\varphi)^a$, defined by the self-adjoint functional calculus, where $\mathrm{Re}\ a > 0$. Note that

$$C_\varphi^* C_\varphi \equiv sC_\sigma C_\varphi \ (\mathrm{mod}\ \mathcal{K}) = sC_{\varphi\circ\sigma} = sC_{\rho_{\eta,2b}}$$

where the last equality follows from Theorem 6. In Theorem 3, take $\gamma = \eta$ and consider the $*-$isomorphism Γ, here called Γ_η to emphasize the fixed point η. We have

$$[(C_\varphi^* C_\varphi)^a] = [C_\varphi^* C_\varphi]^a = s^a [C_{\rho_{\eta,2b}}]^a = s^a \Gamma_\eta (t^{2b})^a = s^a \Gamma_\eta (t^{2ba}) = s^a [C_{\rho_{\eta,2ba}}].$$

The first and fourth equalities follow, respectively, from the facts that the coset map $B \mapsto [B]$ and Γ_η are each $*$-homomorphisms. Relabeling $2ba$ as a, we see that $s^{-\frac{a}{2b}} (C_\varphi^* C_\varphi)^{\frac{a}{2b}}$ is a distinguished representative of $[C_{\rho_{\eta,a}}]$ for Re $a > 0$. A similar argument shows that the coset $[C_{\rho_{\zeta,a}}]$ has distinguished representative $s^{-\frac{a}{2c}} (C_\varphi C_\varphi^*)^{\frac{a}{2c}}$ for Re $a > 0$.

Now consider $\rho_{\eta,a} \circ \varphi$, which we know to be a self-map of $\mathbb{D}$, but not an automorphism, when Re $a > -b$. First we look at the case Re $a > 0$. We have

$$C_{\rho_{\eta,a} \circ \varphi} = C_\varphi C_{\rho_{\eta,a}} = U (C_\varphi^* C_\varphi)^{\frac{1}{2}} C_{\rho_{\eta,a}} \equiv s^{-\frac{a}{2b}} U (C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2b}} \pmod{\mathcal{K}}$$

by our above discussion. By the spectral theorem, $(C_\varphi^* C_\varphi)^z$ is holomorphic for Re $z > 0$ in the weak operator topology, and therefore in the operator norm topology; see [11], Theorem 3.10.1. Thus the cosets $[C_{\rho_{\eta,a} \circ \varphi}]$ and $[s^{-\frac{a}{2b}} U (C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2b}}]$ are both holomorphic $\mathcal{B}(H^2)/\mathcal{K}$-valued functions of a, Re $a > -b$, which agree on the subset $\{a : \text{Re } a > 0\}$. Hence they agree on all of $\{a : \text{Re } a > -b\}$, showing that $s^{-\frac{a}{2b}} U (C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2b}}$ is a distinguished representative of $[C_{\rho_{\eta,a} \circ \varphi}]$ when Re $a > -b$. An analogous statement holds for $[C_{\rho_{\zeta,a} \circ \sigma}]$ with Re $a > -c$. The following table summarizes these conclusions.

Table II

Linear-fractional ψ with C_ψ in $C^*(C_\varphi, \mathcal{K})$

Condition on ψ in Theorem 5	ψ	Distinguished representative of $[C_\psi]$	Matrix function $\Psi(C_\psi)(t)$, $0 \le t \le s$
(d)	$\rho_{\eta,a}$, Re $a > 0$	$s^{-\frac{a}{2b}} (C_\varphi^* C_\varphi)^{\frac{a}{2b}}$	$\begin{bmatrix} \left(\frac{t}{s}\right)^{\frac{a}{2b}} & 0 \\ 0 & 0 \end{bmatrix}$
(b)	$\rho_{\zeta,a}$, Re $a > 0$	$s^{-\frac{a}{2c}} (C_\varphi C_\varphi^*)^{\frac{a}{2c}}$	$\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{t}{s}\right)^{\frac{a}{2c}} \end{bmatrix}$
(a)	$\rho_{\eta,a} \circ \varphi$, Re $a > -b$	$s^{-\frac{a}{2b}} U (C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2b}}$	$\begin{bmatrix} 0 & \sqrt{t} \left(\frac{t}{s}\right)^{\frac{a}{2b}} \\ 0 & 0 \end{bmatrix}$
(c)	$\rho_{\zeta,a} \circ \sigma$, Re $a > -c$	$s^{-\frac{a}{2c} - 1} U^* (C_\varphi C_\varphi^*)^{\frac{1}{2} + \frac{a}{2c}}$	$\begin{bmatrix} 0 & 0 \\ \frac{\sqrt{t}}{s} \left(\frac{t}{s}\right)^{\frac{a}{2c}} & 0 \end{bmatrix}$

Given an operator B in $C^*(C_\varphi, \mathcal{K})$, $\sigma_e(B)$ and $\|B\|_e$ coincide with $\sigma(\Psi(B))$ and $\|\Psi(B)\|$, respectively. Thus, if B is a finite linear combination of composition operators C_ψ with ψ's chosen from Column 2 in Table II, one can calculate $\Psi(B)$ from Column 4 and in principle read off $\sigma_e(B)$ and $\|B\|_e$; see Theorem 4.17 in [15].

It is known [14] that the collection of linear-fractional composition operators C_ψ with ψ a non-automorphism having $\|\psi\|_\infty = 1$ is linearly independent modulo $\mathcal{K}$. The following result shows that this remains true when $\mathcal{K}$ is replaced by the larger subspace $C^*(C_\varphi, \mathcal{K})$ of $\mathcal{B}(H^2)$.

Theorem 7. *Let φ be as in (1). Suppose that $\beta_1, \cdots, \beta_q$ are distinct linear-fractional self-maps of $\mathbb{D}$ and that $a_1 \cdots, a_q$ are non-zero complex numbers. If the linear combination $a_1 C_{\beta_1} + \cdots + a_q C_{\beta_q}$ lies in $C^*(C_\varphi, \mathcal{K})$, then so do the individual operators $C_{\beta_1}, \cdots, C_{\beta_q}$.*

Proof. Let us discard those C_{β_i}'s which lie in $C^*(C_\varphi, \mathcal{K})$ and assume for the purpose of obtaining a contradiction that there are some left over. Relabel these as $C_{\beta_1}, \cdots, C_{\beta_r}$, let $a_1, \cdots, a_r$ be the corresponding constants, and put $T = a_1 C_{\beta_1} + \cdots + a_r C_{\beta_r}$, which lies in $C^*(C_\varphi, \mathcal{K})$. Here, none of the summands are compact, so $\|\beta_i\|_\infty = 1$ for $i = 1, \cdots, r$. Now we proceed almost as in the proof of Theorem 1, with T playing the role of C_ψ. Given $\epsilon > 0$ there exists A as in that proof and a complex c such that $\|T - cC_z - A\|_e < \epsilon$. By the inequality (11)

$$\|T - cC_z - A\|_e^2 \geq |c|^2 + \frac{1}{2\pi} \sum_{i=1}^{r} |a_i|^2 |J(\beta_i)|$$

so that $|c| < \epsilon$, since each $|J(\beta_i)|$ must be zero, so β_i is a non-automorphism. As earlier, we have $\|T - A\|_e < 2\epsilon$.

According to Corollary 5.17 in [14], the cosets $[C_{\beta_1}], \cdots, [C_{\beta_r}]$ are linearly independent in $\mathcal{B}(H^2)/\mathcal{K}$. It follows that T is not compact, so the matrix function

$$\Psi(T) = \begin{bmatrix} f_2 & f_3 \\ f_4 & f_1 \end{bmatrix} \tag{23}$$

is not identically zero on $[0, s]$. As in Theorem 1, we focus on f_3 and aim for a contradiction by assuming that $\|f_3\|_\infty > 0$. An appropriate choice of ϵ again yields the inequality (12), where p has the same meaning as there. Again we write A in the form (13), and thus have

$$\begin{aligned} 4\epsilon^2 > \|T - A\|_e^2 &\geq \limsup_{|z| \to 1} \left\| (T^* - A^*) \frac{k_z}{\|k_z\|} \right\|^2 \\ &\geq \lim_{\Gamma_{\zeta, D}} \left\| \left(\sum_{i=1}^{r} \overline{a_i} C_{\beta_i}^* - \sum_{i=1}^{m} \overline{c_i} C_{\psi_i}^* \right) \frac{k_z}{\|k_z\|} \right\|^2 \\ &\geq \left\| \sum_{\substack{\zeta \in F(\psi_i) \\ D_1(\psi_i, \zeta) = \mathbf{d_3}}} \overline{c_i} k_{w_i}^+ \right\|_{H_+^2}^2 \end{aligned}$$

The rest of the argument follows that of Theorem 1 exactly, reaching the same contradiction. □

5. Non linear-fractional maps

In this section we explore maps ψ, satisfying either condition (e) or (f) of Theorem 2, for which C_ψ lies in $C^*(C_\varphi, \mathcal{K})$; our main result shows that such maps exist. We begin with a lemma about finite Blaschke products.

Lemma 5. *Given ζ, η distinct points on $\partial\mathbb{D}$, and positive numbers t_1, t_2, there exists a finite Blaschke product B with the properties $B(\eta) = \eta, B(\zeta) = \eta, B'(\eta) = t_1$ and $|B'(\zeta)| = t_2$. Moreover, $B'(\zeta) = \eta\overline{\zeta} t_2$.*

Proof. Clearly there is no loss of generality in taking $\eta = 1$. Initially we will also suppose that $\zeta = -1$; this condition will be removed at the end. A finite Blaschke product $B(z) = \prod \frac{|a_n|}{a_n}(a_n - z)/(1 - \overline{a_n}z)$ will meet the conditions $B(1) = 1$, $B(-1) = 1$ if both

$$\prod \frac{|a_n|}{a_n}\frac{a_n - 1}{1 - \overline{a_n}} = 1$$

and

$$\prod \frac{|a_n|}{a_n}\frac{a_n + 1}{1 + \overline{a_n}} = 1.$$

It is easy to see that both of these conditions will be met if the zeros of B are chosen to be a collection of conjugate pairs $\{a, \overline{a}\}$. The conditions $B'(1) = t_1, |B'(-1)| = t_2$ are satisfied if

$$\sum \frac{1 - |a_n|^2}{|1 - a_n|^2} = t_1 \tag{24}$$

and

$$\sum \frac{1 - |a_n|^2}{|1 + a_n|^2} = t_2 \tag{25}$$

respectively (see [2]).

Next observe that for any $t > 0$, $\{z : 1 - |z|^2 = t|1 - z|^2\}$ is a circle centered at $(t/(t+1), 0)$ with radius $1/(t+1)$ and $\{z : 1 - |z|^2 = t|1 + z|^2\}$ is a circle centered at $(-t/(t+1), 0)$ with radius $1/(1+t)$. As $t \to 0$, the centers of these circles approach 0 and the radii tend to 1. Thus given t_1, t_2 arbitrary positive numbers we may choose m a positive integer sufficiently large so that the circles $\{z : 1 - |z|^2 = \frac{t_1}{2^m}|1 - z|^2\}$ and $\{z : 1 - |z|^2 = \frac{t_2}{2^m}|1 + z|^2\}$ intersect in a conjugate pair of points $a, \overline{a}$. Consider the Blaschke product $B(z)$ with a zero of order 2^{m-1} at a and a zero of order 2^{m-1} at $\overline{a}$. Since the zeros occur at conjugate pairs, $B(1) = 1$ and $B(-1) = 1$. By construction

$$(1 - |a|^2)/(|1 - a|^2) = (1 - |\overline{a}|^2)/(|1 - \overline{a}|^2) = \frac{t_1}{2^m},$$

and

$$(1 - |a|^2)/(|1 + a|^2) = (1 - |\overline{a}|^2)/(|1 + \overline{a}|^2) = \frac{t_2}{2^m},$$

so that the zeros of B satisfy Equations (24) and (25) as desired, and $B'(1) = t_1$, $|B'(-1)| = t_2$.

Now suppose $\zeta \in \partial\mathbb{D}$ is not equal to -1. Find a parabolic automorphism τ fixing 1, with derivative 1 there, and taking ζ to -1; a unique such τ exists since (purely imaginary) translations act transitively on the boundary of the right half-plane. Then for B as constructed above, $B \circ \tau$ is a finite Blaschke product fixing 1, with derivative t_1 at 1, sending ζ to 1 and having derivative $|(B\circ\tau)'(\zeta)| = |B'(-1)||\tau'(\zeta)|$; since $|B'(-1)|$ can be arbitarity prescribed and τ depends only on the value of ζ, this means $|(B \circ \tau)'(\zeta)|$ can be chosen to be an arbitrary positive number. Finally observe that if B is a Blaschke product with $B(\zeta) = \eta$ and $|B'(\zeta)| = s$, then we must have $B'(\zeta) = \eta\overline{\zeta}s$, since $\overline{\eta}\zeta B(z)$ fixes ζ, and hence has positive derivative there. $\square$

Theorem 8. *Suppose that φ is as in (1). There exist analytic self-maps ψ_1 and ψ_2 of $\mathbb{D}$, satisfying conditions (e) and (f) of Theorem 2 respectively, such that C_{ψ_1} and*

C_{ψ_2} lie in $C^*(C_\varphi, \mathcal{K})$. Moreover, ψ_1 and ψ_2 can be taken to extend continuously to the closed disk $\overline{\mathbb{D}}$.

Proof. We shall indicate the construction for ψ_1 with the normalization $\eta = 1$. First consider a simply connected domain Ω in $\mathbb{D}$, whose boundary is a smooth Jordan curve which at 1 and at -1 includes an arc of an internally tangent circle, such that $\overline{\Omega} \cap \partial\mathbb{D} = \{-1, 1\}$. It is easy to construct a conformal map $\rho : \mathbb{D}$ onto Ω, extending to a homeomorphism of $\overline{D}$ onto $\overline{\Omega}$, with the properties $\rho(1) = 1, \rho(-1) = -1$ and $\rho'(1) = \frac{1}{2}$. The map ρ will be analytic in a neighborhood of both 1 and -1. Let τ be the unique parabolic automorphism of $\mathbb{D}$ fixing 1, having derivative 1 there, and mapping ζ to -1. By the uniqueness statement, $\tau'(\zeta)$ is determined by ζ. Using Lemma 5 construct a finite Blaschke product B with $B(1) = 1, B(-1) = 1, B'(1) = 2$ and $|B'(-1)|$ chosen so that $|B'(-1)||\rho'(-1)||\tau'(\zeta)| = s$, where $s = |\varphi'(\zeta)|$. The map $\psi_1 \equiv B \circ \rho \circ \tau$ is a self-map of $\mathbb{D}$, with finite angular derivative set $F(\psi_1) = \{\zeta, 1\}$ and satisfying $\psi_1(\zeta) = 1, \psi_1(1) = 1, \psi_1'(1) = 1$ and $\psi_1'(\zeta) = \overline{\zeta}s$, this last condition following from $|\psi_1'(\zeta)| = s$ and $\psi_1(\zeta) = 1$. Clearly ψ_1 has order of contact two at 1 and ζ.

Any linear-fractional map β is uniquely determined by its second order data vector $D_2(\beta, z_0) = (\beta(z_0), \beta'(z_0), \beta''(z_0))$ at any point z_0 of analyticity. The curvature of the curve $\psi_1(\partial\mathbb{D})$ at the points $\psi_1(\zeta)$ and $\psi_1(1)$ is determined by $D_2(\psi_1, \zeta)$ and $D_2(\psi_1, 1)$, respectively. By construction of ψ_1, these curvatures exceed unity. There exists unique linear-fractional maps β_1, β_2 of $\mathbb{D}$ with the second order data vectors

$$D_2(\beta_1, 1) = D_2(\psi_1, 1) = (1, 1, \psi_1''(1)) \text{ and } D_2(\beta_2, \zeta) = D_2(\psi_1, \zeta) = (1, \overline{\zeta}s, \psi_1''(\zeta)).$$

The curvature of $\{\psi_1(e^{i\theta}) : e^{i\theta} \in \partial\mathbb{D}\}$ matches that of $\beta_1(\partial\mathbb{D})$ at $e^{i\theta} = 1$ and that of $\beta_2(\partial\mathbb{D})$ at $e^{i\theta} = \zeta$. Thus β_1 and β_2 are non-automorphism self-maps of $\mathbb{D}$. Since $\varphi'(\zeta) = \psi_1'(\zeta) = \beta_2'(\zeta)$, Theorem 5 shows that C_{β_1} and C_{β_2} are in $C^*(C_\varphi, \mathcal{K})$, and hence so is $C_{\beta_1} + C_{\beta_2}$. By Corollary 5.16 of [14], $C_{\psi_1} \equiv C_{\beta_1} + C_{\beta_2} \pmod{\mathcal{K}}$ and thus C_{ψ_1} is in $C^*(C_\varphi, \mathcal{K})$.

For ψ_2, note that the Krein adjoint σ of φ satisfies $\sigma(\eta) = \zeta$ and $\sigma'(\eta) = 1/\varphi'(\zeta)$. We apply the first part of the proof, with φ replaced by σ, to find a self map ψ_2 of $\mathbb{D}$ with $F(\psi_2) = \{\zeta, \eta\}$, $\psi_2(\eta) = \zeta$, $\psi_2'(\eta) = \sigma'(\eta) = 1/\varphi'(\zeta)$, $\psi_2(\zeta) = \zeta$, and $\psi_2'(\zeta) = 1$ such that $C_{\psi_2} \in C^*(C_\sigma, \mathcal{K}) = C^*(C_\varphi, \mathcal{K})$ as desired. □

Our last theorem shows that for sufficiently nice ψ, Theorem 2 is almost the whole story.

Theorem 9. *Let ψ be an analytic self-map of $\mathbb{D}$ such that $F(\psi)$ is a finite set, ψ extends analytically to a neighborhood of each point in $F(\psi)$, and for any open set U of $\partial\mathbb{D}$ containing $F(\psi)$, $\|\chi_{\partial\mathbb{D}\setminus U}\psi\|_\infty < 1$. If φ is as in (1), then C_ψ lies in $C^*(C_\varphi, \mathcal{K})$ if and only if*

(i) *one of the conditions (a)-(f) of Theorem 2 holds, and*
(ii) *the map ψ has order of contact two at each point of $F(\psi)$.*

Proof. Suppose ψ is as described and C_ψ is in $C^*(C_\varphi, \mathcal{K})$. By Theorem 2, one of (a)-(f) holds. Suppose that γ is in $F(\psi)$ and ψ has order of contact exceeding two at γ. Following Theorem 1, we let $\epsilon > 0$ and find a finite linear combination A of composition operators whose self-maps are chosen from the lists (6) such that $\|C_\psi - A\| < \epsilon$. At the same time we apply the inequality (9) to the linear

combination $C_\psi - A$ at the point $\alpha = \gamma$. The maps in the lists (6), being linear-fractional non-automorphisms, all have order of contact two at the unique points in their angular derivative sets. Taking the left side of (9) to be $\|C_\psi - A\|_e^2$ and $k = 3$ on the right side, the sum on the right-hand side has only one term, namely $1/|\psi'(\gamma)|$, giving

$$\epsilon^2 > \|C_\psi - A\|_e^2 \geq \frac{1}{|\psi'(\gamma)|},$$

a contradiction. Thus ψ must have order of contact two at γ.

Conversely, suppose ψ satisfies (i) and (ii). If $F(\psi) = \{\zeta\}$, let β be the unique linear-fractional map with $D_2(\beta, \zeta) = D_2(\psi, \zeta)$. Since ψ has order of contact two at ζ, the curvature of $\{\psi(e^{i\theta}) : e^{i\theta} \in \partial\mathbb{D}\}$ at $e^{i\theta} = \zeta$ exceeds unity, so that $\beta(\partial\mathbb{D})$, having the same curvature, is internally tangent to $\partial\mathbb{D}$ at $\psi(\zeta)$ and bounds a proper subdisk of $\mathbb{D}$. Thus β is a non-automorphism of $\mathbb{D}$ satisfying (a) or (b) of Theorem 5, and C_β lies in $C^*(C_\varphi, \mathcal{K})$. The same argument covers the case $F(\psi) = \{\eta\}$.

If $F(\psi) = \{\zeta, \eta\}$ we proceed as in the proof of Theorem 8 to produce linear-fractional non-automorphisms β_1 and β_2 of $\mathbb{D}$ with $C^*(C_\varphi, \mathcal{K})$ containing C_{β_1} and C_{β_2} and $C_\psi \equiv C_{\beta_1} + C_{\beta_2}$ (mod $\mathcal{K}$). □

References

[1] P. Bourdon, D. Levi, S. Narayan and J. Shapiro, Which linear fractional composition operators are essentially normal?, J. Math. Anal. Appl. 280 (2003), 30–53.

[2] P. Ahern and D. Clark, On inner functions with H^p-derivative, Michigan Math. J. 21 (1974), 115–127.

[3] E. Basor and D. Retsek, Extremal non-compactness of composition operators with linear fractional symbol, J. Math. Anal. Appl. 322 (2006), 749–763.

[4] E. Berkson, Composition operators isolated in the uniform operator topology, Proc. Amer. Math. Soc. 81 (1981), 230–232.

[5] J. Conway, *A Course in Operator Theory*, Amer. Math. Soc., Providence, 2000.

[6] C. Cowen, Composition operators on H^2, J. Operator Theory 9 (1983), 77–106.

[7] C. Cowen, Linear-fractional composition operators on H^2, Integral Equations Operator Theory 11 (1988), 151–160.

[8] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[9] A. Dash, Joint essential spectra, Pacific J. Math., 64 (1976), 119–128.

[10] J. Guyker, On reducing subspaces of composition operators, Acta Sci. Math. (Szeged) 53 (1989), 369–376.

[11] E. Hille and R. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc., Providence, 1957.

[12] M. Jury, C^*-algebras generated by groups of composition operators, Indiana Univ. Math. J. 56 (2007), 3171–3192.

[13] M. Jury, The Fredholm index for elements of Toeplitz-composition C^*-algebras, Integral Equations Operator Theory, 58 (2007), 341–362.

[14] T. Kriete and J. Moorhouse, Linear relations in the Calkin algebra for composition operators, Trans. Amer. Math. Soc. 359 (2007), 2915–2944.

[15] T. Kriete, B. MacCluer, and J. Moorhouse, Toeplitz-composition C^*-algebras, J. Operator Theory 58 (2007), 135–156.

[16] T. Kriete, B. MacCluer, and J. Moorhouse, Spectral theory for algebraic combinations of Toeplitz and composition operators, preprint.

[17] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canadian J. Math. 38 (1986), 878–906.

[18] A. Montes-Rodríguez, M. Ponce-Escudero, and S. Shkarin, Invariant subspaces of parabolic self-maps in the Hardy space, preprint.

[19] J. Shapiro and C. Sundberg, Isolation amongst the composition operators, Pacific J. Math. 145 (1990), 117–152.

[20] J. Shapiro and P. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators on H^2, Indiana Univ. Math. J. 23 (1973), 471–496.

[21] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

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